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ON THE EXPANSION OF A FUNCTION WITHOUT USE OF DERIVATIVES.*

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1. Let $F(x)$ be a uniform, finite, and continuous function throughout the interval from $x = a$ to $x = c$, inclusive. If the successive derivatives are also uniform, finite, and continuous, the function will be called a Taylor's function when it can be expressed by Taylor's series throughout the interval (ac) . For the present we consider $F(x)$ to be a Taylor's function.

Let

$$\begin{aligned}\int_1 F(x) &= \int_a^x F(x) dx, \\ \int_2 F(x) &= \int_a^x dx \int_a^x F(x) dx \\ &= \int_a^x dx \int_1 F(x),\end{aligned}$$

and generally

$$\int_n F(x) = \int_a^x dx \int_{n-1} F(x),$$

be called the successive definite integrals of $F(x)$ with respect to the base a , or briefly the successive integrals of $F(x)$.

2. If the function $F(x)$ and its first n successive integrals vanish at $x = c$, then must the function $F(x)$ vanish $n + 1$ distinct times between a and c , inclusive of c . For,

$$\begin{aligned}\int_n F(c) &= \int_a^c dx \int_{n-1} F(x) \\ &= (c - a) \int_{n-1} F(u_1) \qquad a < u_1 < c \\ &= 0.\end{aligned}$$

Therefore

$$\int_{n-1} F(x) = 0,$$

for $x = u_1$. But

$$\int_{n-1} F(x) = 0,$$

* Read before American Mathematical Society, at Springfield, Mass., Aug. 27, 1895.

for $x = c$, and

$$\int_{n-1} F(x) = \int_a^{u_1} dx \int_{n-2} F(x) + \int_{u_1}^c dx \int_{n-2} F(x).$$

Also

$$\begin{aligned} \int_{n-1} F(u_1) &= \int_a^{u_1} dx \int_{n-2} F(x) \\ &= (u_1 - a) \int_{n-2} F(u_2) \quad a < u_2 < u_1 \\ &= 0. \end{aligned}$$

Consequently,

$$\int_{n-2} F(x) = 0,$$

for $x = u_2$, and therefore must

$$\begin{aligned} \int_{u_1}^c dx \int_{n-2} F(x) &= (c - u_1) \int_{n-2} F(u_3) \quad u_1 < u_3 < c \\ &= 0, \end{aligned}$$

and we have

$$\int_{n-2} F(x) = 0,$$

for the three values of x , viz. u_2 , u_3 , c .

Proceeding in like manner, we show that

$$\int_{n-r} F(x) = 0,$$

for $r + 1$ values of x between a and c , including c ; and finally that

$$F(x) = 0,$$

for $n + 1$ values $u_1, u_2, \dots, u_n, c$ of x in the interval (ac) .

3. By a theorem of the Differential Calculus,* when $F(x)$ is a Taylor's function in (ac) , we have

$$F(x) = (x - u_1) \dots (x - u_n) (c - x) \frac{F^{n+1}(u)}{(n + 1)!};$$

and since we may write $x - u_r = \theta_r (c - a)$, and $c - x = \theta' (c - a)$, wherein the absolute values of θ_r, θ' lie between zero and unity, we have

$$F(x) = \theta^{n+1} (c - a)^{n+1} \frac{F^{n+1}(u)}{(n + 1)!},$$

wherein $0 < \theta < 1$ and u lies in the interval (ac) .

* See Laurent, *Traité d'Analyse*, t. i; or *Annals of Mathematics*, vol. viii, p. 74.

4. The function

$$F(x) = \frac{\begin{vmatrix} f(x), & \varphi_1(x), & \dots, & \varphi_{n+1}(x) \\ f(c), & \varphi_1(c), & \dots, & \varphi_{n+1}(c) \\ \int_1 f(c), & \int_1 \varphi_1(c), & \dots, & \int_1 \varphi_{n+1}(c) \\ \dots & \dots & \dots & \dots \\ \int_n f(c), & \int_n \varphi_1(c), & \dots, & \int_n \varphi_{n+1}(c) \end{vmatrix}}{\begin{vmatrix} \varphi_1(c), & \int_1 \varphi_2(c), & \dots, & \int_n \varphi_{n+1}(c) \end{vmatrix}}$$

vanishes, as well as do its first n successive integrals at $x = c$. Consequently, if the φ -functions are Taylor functions also,

$$f(x) = \sum_0^{n+1} A_r \varphi_r(x) + \theta^{n+1} \frac{(a-c)^{n+1}}{(n+1)!} F^{n+1}(u),$$

in which the coefficients of the φ -functions contain only definite integrals.

5. Let

$$\varphi_{r+1}(x) = \frac{1}{r!} \left[\frac{x-a}{c-a} \right]^r.$$

Then,

$$F^{n+1}(u) = f^{n+1}(u),$$

and we have

$$\begin{vmatrix} f(x), & 1, & \frac{1}{1!} \left[\frac{x-a}{c-a} \right], & \frac{1}{2!} \left[\frac{x-a}{c-a} \right]^2, & \dots, & \frac{1}{n!} \left[\frac{x-a}{c-a} \right]^n \\ f(c), & \frac{1}{0!}, & \frac{1}{1!}, & \frac{1}{2!}, & \dots, & \frac{1}{n!} \\ \frac{1}{c-a} \int_1 f(c), & \frac{1}{1!}, & \frac{1}{2!}, & \frac{1}{3!}, & \dots, & \frac{1}{(n+1)!} \\ \frac{1}{(c-a)^2} \int_2 f(c), & \frac{1}{2!}, & \frac{1}{3!}, & \frac{1}{4!}, & \dots, & \frac{1}{(n+2)!} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(c-a)^n} \int_n f(c), & \frac{1}{n!}, & \frac{1}{(n+1)!}, & \frac{1}{(n+2)!}, & \dots, & \frac{1}{(2n)!} \end{vmatrix} = \frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u),$$

$$\left| \frac{1}{0!}, \frac{1}{2!}, \frac{1}{4!}, \dots, \frac{1}{(2n)!} \right|$$

or

$$f(x) \sum_{r=0}^n A_r \frac{(-1)^r}{r!} \left[\frac{x-a}{c-a} \right]^r + \frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u), \quad (\text{i})$$

or

$$f(x) = \sum_{r=0}^n A_r \frac{(-1)^r}{(c-a)^r} \int_r f(c) + \frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u), \quad (\text{ii})$$

wherein

$$A_r = \sum_{p=0}^n (-1)^p \frac{\Delta(r, p)}{\Delta} \frac{\int_p f(c)}{(c-a)^p}, \quad \text{in (i)}$$

or

$$A_r = \sum_{p=0}^n (-1)^p \frac{\Delta(r, p)}{\Delta} \frac{1}{p!} \left[\frac{x-a}{c-a} \right]^p; \quad \text{in (ii)}$$

and $\Delta(m, n)$ is the per-symmetric determinant

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0! & 2! & \cdots & (2n)! \end{vmatrix},$$

with its m th column and n th row deleted.

Since

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots,$$

we have

$$\frac{1}{(c-a)^r} \int_r f(c) = \frac{1}{r!} \left\{ f(a) + \frac{c-a}{1!} \frac{f'(a)}{r+1} + \frac{(c-a)^2}{2!} \frac{f''(a)}{(r+1)(r+2)} + \cdots \right\},$$

which becomes evanescent as r increases without limit, as does also

$$\frac{(c-a)^{n+1}}{(n+1)!} \theta^{n+1} f^{n+1}(u)$$

when $n = \infty$.

The series

$$\sum_{r=0}^{\infty} \frac{1}{r!} \left[\frac{x-a}{c-a} \right]^r \quad \text{and} \quad \sum_{r=0}^{\infty} \frac{1}{(c-a)^r} \int_r f(c)$$

are absolutely convergent. If the number $[\Delta(r, p)/\Delta]_{n=\infty}$ has a finite limit, the series (i) and (ii) are convergent infinite series. Unfortunately, I have not, as yet, been able to compute these numbers, and thus put the series in practicable form.

6. Under the assumption that $[\Delta(r, p)/\Delta]_{n=\infty}$ is finite, we have

$$f(x) = \sum_{r=0}^{\infty} A_r \frac{(-1)^r}{r!} \left[\frac{x-a}{c-a} \right]^r \quad (\text{iii})$$

throughout the interval (ac) , and

$$f^r(a) = \frac{A_r}{(a - c)^r}$$

for $f(x)$ any Taylor's function.

The coefficients A , nowhere involve derivatives, but only numbers and definite integrals. If $f(x)$ be uniform, finite, and continuous, but derivativeless throughout (ac) , its successive integrals still obtain, and the numerical equivalence between the function and the series (iii) still must hold good. But the series of integral powers of $(x - a)$ represents a Taylor's function having the same *arithmetical* locus as the function $f(x)$ throughout (ac) . The result, geometrically interpreted, is that we have drawn a Taylor's function curve having the same geometrical locus as the analytical function $f(x)$ throughout (ac) , and the tangent to the Taylor's curve at any point x , represents the geometrical direction of the analytical curve there, which direction is analytically indeterminate.*

UNIVERSITY OF VIRGINIA, *April*, 1895.

* The possibility of the assumption that a Taylor's function could have the same geometrical locus as that of a uniform, finite, continuous, and derivativeless function, was not approved of at the reading of this note before the Society.